A CATEGORY OF GROUPOIDS WITH MULTIPLICATION-COMMUTING FUZZY ACTIONS

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ABSTRACT: *This paper introduces a new category whose objects are groupoids and whose morphisms are multiplication-commuting Q-fuzzy actions, where Q is a unital quantale. It is shown that each such morphism can be equivalently regarded as a Q-fuzzy subgroupoid of a suitably constructed groupoid, providing a unified structural perspective on Q-graded fuzzy actions. Furthermore, the paper proves that the newly defined category is isomorphic to a certain subcategory of Q-Rel, thereby establishing a natural correspondence between the introduced morphisms and Q-fuzzy relations. These results build a conceptual bridge between algebraic and relational frameworks within a unital quantale setting.*

KEY WORDS: *groupoid, quantale, Q- fuzzy groupoid, Q-fuzzy action, multiplication-commuting Q-fuzzy action, Q-Rel*

1. INTRODUCTION

Category theory offers a framework in which relationships, transformations, and abstractions can be expressed in a coherent and highly general manner ("an abstract setting for comparison and analogy" [2]). In this spirit, the present paper introduces a new categorical construction whose objects are groupoids and whose arrows are fuzzy actions, a generalization of groupoid actions that incorporates degrees of membership and uncertainty, as modeled in fuzzy set theory. Classical groupoid actions provide a powerful mechanism for encoding symmetry and local equivalence, yet they rely on exact, binary relations. Many real-world and theoretical contexts—such as systems with approximate symmetries, uncertain dynamics, or partial compatibilities—demand a framework that can interpolate between fully determined and indeterminate interactions. Fuzzy actions naturally extend groupoid actions to this broader setting, allowing morphisms to encode degrees of compatibility between structures rather than strict homomorphisms.

Concerning the groupoids (small categories with inverses), we use the same notation as in [5-8].

A quantale [13] $(Q, \leq, *)$ is a complete lattice $(Q, \leq)$ equipped with an associative binary operation $*: Q \times Q \to Q$ that distributes over arbitrary joins (suprema). If $*$ is also commutative, then Q is called a commutative quantale. When $*$ has a unit element e, the quantale is said to be unital. In this paper, the greatest lower bound (infimum) of a subset S $\subset$ Q is denoted by inf S, and the least upper bound (supremum) by sup S. The bottom element of Q is denoted by 0. A unital quantale is called integral if its unit element e coincides with the top element of the lattice $(Q, \leq)$.

If $*$ is a left-continuous triangular norm (t-norm) on a complete lattice $(L, \leq)$ (see, for example, [12]), then $(L. \leq, *)$, is a commutative integral quantale, where $\leq$ is the usual order. If $\{0, 1\}$ is endowed with the usual order $\leq$ and $*$is an arbitrary t-norm. then $(\{0, 1\}, \leq, *)$, is a commutative integral quantale.

In this paper, we consider certain a commutative unital quantale $(Q, \leq, *)$. For a, b $\in$ Q, we write $a \geq b$ iff $b \leq a$

2. THE NOTION OF MULTIPLICATION - COMMUTING FUZZY ACTION

Let G be a groupoid and H a set. Consider the set $H \times G \times H$. This set can be endowed with a groupoid structure where, the composition of two elements is defined as

$$(x, g_1, y)(w, g_2, z) = (x, g_1g_2, z),$$

whenever $d(g_1) = r(g_2)$ and $y = w$, while the inverse of an element is given by

$$(x, g, y)^{-1} = (x, g^{-1}, y).$$

For the groupoid $X \times G \times X$, the range and source maps are defined as follows:
$r(x, g, y) = (x, r(g), x)$, $d(x, g, y) = (y, d(g), y)$, where $r(g)$ and $d(g)$ denote the range and source of g in G, respectively.
Therefore, the unit space of $X \times G \times X$ can be identified with $X \times G^{(0)}$ via the mapping:

$$(x, u, x) \mapsto (x, u).$$

If $\sigma: X \to G^{(0)}$ is a map, then the standard blow-up groupoid $G[X, \sigma]$ of G associated with σ is a subgroupoid of $X \times G \times X$. Specifically,

$$G[X, \sigma] = \{(x, g, y) \in X \times G \times X, \ \sigma(x) = r(g), \sigma(y) = d(g)\},$$

If $\rho : X \to S$ is a map, then we denote by

$$G[X, \sigma, \rho] = \{(x, g, y) \in X \times G \times X, \ \sigma(x) = r(g), \sigma(y) = d(g), \rho(x) = \rho(y)\}.$$

It is easy to see that $G[X, \sigma, \rho]$ is a subgroupoid of $G[X, \sigma]$.

The starting point in the definition of the notion of multiplication - commuting fuzzy action is the concept of a morphism introduced in [9] through the reformulation of the notion of a morphism in [16] and [17] in terms of groupoid actions. A detailed description of the connection between Zakrzewski morphisms and groupoid actions, along with some interesting examples, can be found in [14]. Let us recall the notion of a morphism introduced in [9] transposed to the right in this paper: by a crisp morphism from a groupoid G to a groupoid H, we understand a right action of G on H that commutes with the left multiplication on H. More precisely, a morphism [9] from a groupoid G to a groupoid H is given by a map

$$\sigma : H^{(0)} \to G^{(0)}$$

together with and a map

$$(x, g) \mapsto x \cdot g$$

from

$$\{(h, g) \in H \times G : \sigma(d(h)) = r(g)\}$$

to H satisfying the following conditions:
1. $\sigma(d(h \cdot g)) = d(g)$ for all $g \in G$ and $h \in H$ such that $r(g) = \sigma(d(h))$.
2. $h \cdot \sigma(d(h)) = h$ for all $h \in H$.
3. If $(g_1, g_2) \in G^{(2)}$, $h \in H$ and $r(g_1) = \sigma(d(h))$, then $h\cdot (g_1g_2) = (h \cdot g_1) \cdot g_2$.
4. $r(h \cdot g) = r(h)$ for all $g \in G$ and $h \in H$ such that $r(g) = \sigma(d(h))$.
5. If $(h_1, h_2) \in H^{(2)}$, $g \in G$ and $r(g) = \sigma(d(h_2))$, then $(h_1h_2) \cdot g = h_1 (h_2 \cdot g_2)$.

For a morphism as above, if

$$H \rtimes G = \{(h, g, h \cdot g) \in H \times G \times H, \ r(g) = \sigma(d(h))\},$$

then $H \rtimes G \subset G[H, \sigma\circ d, r] \subset G[H, \sigma\circ d] \subset H \times G \times H$ (inclusions of subgroupoids).

Therefore, a natural definition for a multiplication - commuting fuzzy (right) action would be as in [6] as a fuzzy subgroupoid (in the sense of [5]) of $G[H, \sigma\circ d, r]$ satisfying a multiplication – commuting condition. In [7] we reformulated the definition of a T-fuzzy groupoid (with T denoting a t-norm) removing certain restrictions concerning the behavior of membership function on the unit space. This modification enables the application of fuzzification techniques to groupoid contractions.. We maintain the same perspective in this paper, substituting the t-norm T with the quantale Q operation ∗.

Definition 2.1. A Q-fuzzy subgroupoid of a groupoid G is a function $\gamma : G \to Q$ such that
1. $\gamma(g_1g_2) \geq \gamma(g_1) * \gamma(g_2)$ for all $(g_1, g_2) \in G^{(2)}$.
2. $\gamma(g^{-1}) \geq \gamma(g)$ for all $g \in G$.

Similarly to the approach in [7] we defined Q-fuzzy equivalence relation on X as Q-fuzzy subgroupoids of $X \times X$.

Definition 2.2. A Q-fuzzy equivalence relation is a function $\xi : X \times X \to Q$ such that
1. $\xi(x, y) \geq \xi(x, z) * \xi(z, y)$ for all $x, y, z \in X$
2. $\xi(x, y) = \xi(y, x)$ for all $x \in X$.

Note that we do not impose $\xi(x, x) = 1$ (see [15], [10]), which typically encodes reflexivity. Instead, this flexibility allows us to consider equivalence relations $E \subset S \times S$ on subsets $S \subset X$. In the classical (crisp) setting, if $x \notin S$, then $(x, x) \notin E$.

This approach facilitates the study of various fuzzy structures, such as fuzzy sets, fuzzy subgroups, fuzzy equivalence relations, and fuzzy group actions [1] within a unified framework.

Definition 2.3. Let G and H be two groupoids. A multiplication - commuting Q - fuzzy (right) action of G on H is given by a map

$$\sigma : H^{(0)} \to G^{(0)}$$

together with and a map

$$\alpha : G[H, \sigma\circ d, r] \to Q$$

satisfying the following conditions:

1. $\alpha(h, g_1g_2, h') \geq \alpha(h, g_1, h'') * \alpha(h'', g_2, h')$ for all $(h, g_1, h''), (h'', g_2, h') \in G[H, \sigma\circ d, r]$.
2. $\alpha(h', g^{-1}, h) \geq \alpha(h, g, h')$ for all (h, g, h') in $G[H, \sigma\circ d, r]$.
3. (multiplication-commuting condition)

$$\alpha(h, g, h') = \alpha(d(h), g, h^{-1}h')$$

for all (h, g, h') in $G[H, \sigma\circ d, r]$.

In the following we denote a multiplication - commuting Q- fuzzy action of G on H as in the preceding definition by

$(\sigma, \alpha) : G \rightharpoonup H$.

The multiplication-commuting condition implies that for $(h, g, h') \in G[H, \sigma\circ d, r]$ and $(h'', h) \in H^{(2)}$ we have:

$$\alpha(h''h, g, h''h') = \alpha(d(h), g, h^{-1} h''^{-1} h''h') = \alpha(d(h), g, h^{-1}h') = \alpha(h, g, h').$$

The crisp condition $h \cdot \sigma(h) = h$ for all h in H, can be encoded requiring that membership function of $\alpha(h, \sigma(h), h) \geq e$. However, we do not adopt this perspective here in order to accommodate groupoid contractions of $G[H, \sigma\circ d, r]$. The multiplication – commuting condition in Definition 2.3 differs from the tentative notion of commuting fuzzy actions proposed in [6], and it is necessary for enabling the composition of fuzzy actions.

The action $\alpha : G[H, \sigma\circ d, r] \to Q$ can be naturally extended to a function defined on $H\times G\times H$, by assigning the value 0 outside the set $G[H, \sigma\circ d, r]$. The extension becomes a Q-fuzzy subgroupoid of $H\times G\times H$.

Proposition 2.4. Let G and H be two groupoids, $(x, g) \mapsto x \cdot g$ be a right (crisp) action of a groupoid G on H that commutes with the left multiplication on H. Let $\sigma: H^{(0)} \to G^{(0)}$ be the map that defines the momentum map of the action. Let

$$\delta : H \times H \to Q$$

be a Q-fuzzy equivalence relation on H satisfying the following invariance property:

$$\delta(h\cdot g, h' \cdot g) = \delta(d(h) \cdot g, h^{-1} h'\cdot g) = \delta(h, h')$$

for all $h, h' \in H$ and $g \in G$ such that $r(h) = r(h')$ and $\sigma(d(h)) = \sigma(d(h')) = r(g)$. If

$$\alpha : G[H, \sigma\circ d, r] \to Q$$

is defined by $\alpha(h, g, h') = \delta(h \cdot g, h')$, then (σ, α) is a multiplication - commuting Q- fuzzy action of G on H.

Proof. In the spirit of Proposition 3.1 [8], since $\delta(h\cdot g, h' \cdot g) = \delta(h, h')$, α is a Q-fuzzy subgroupoid of $G[H, \sigma\circ d, r]$. Therefore, we only need to check the multiplication-commuting condition. We have

$\alpha(h, g, h') = \delta(h \cdot g, h') = \delta(d(h) \cdot g, h^{-1} h'\cdot g) = \alpha(d(h), g, h^{-1}h')$ for all $(h,g,h') \in G[H, \sigma\circ d, r]$.

Consequently, for any morphism in the sense of [9] and any fuzzy equivalence relation invariant under both action and multiplication, a corresponding multiplication - commuting Q -fuzzy action can be assigned. In particular, this allows the association of a multiplication - commuting Q -fuzzy action to any morphism in category studied in [3], whose objects are the groupoids derived in [11] from discrete dynamical systems. Additionally, the perspective in Section 4, according to with a multiplicative - commuting Q -fuzzy action can be viewed as Q-fuzzy subgroupoid of a suitable groupoid, can help clarify the nature of morphisms in [3].

3. THE CATEGORY GrpFAct

Let us define the new category GrpFAct

Objects: groupoids.

Morphisms: multiplication-commuting Q-fuzzy actions

Composition law: Let $(\sigma_{12}, \alpha_{12}) : G_1 \to G_2$ and $(\sigma_{23}, \alpha_{23}) : G_2 \to G_3$ be two multiplication-commuting Q-fuzzy actions (morphisms). Let

$(\sigma_{13}, \alpha_{13}) = (\sigma_{23}, \alpha_{23}) \circ (\sigma_{12}, \alpha_{12}) : G_1 \to G_3$

be defined by

$\sigma_{13} = \sigma_{12} \circ \sigma_{23}$ (composition of maps).

$\alpha_{13} : G_1[G_3, \sigma_{13}\circ d, r] \to Q$, defined by

$\alpha_{13}(s, x, s') =$

$= \sup \{\alpha_{12}(\sigma_{23}(d(s)), x, t) * \alpha_{23}(s, t, s'), t \in G_2$ such that $(s, t, s') \in G_2[G_3, \sigma_{23}\circ d, r]\}$,

for all $(s, x, s') \in G_1[G_3, \sigma_{13}\circ d, r]$.

Proposition 3. 1. Let $(\sigma_{12}, \alpha_{12}) : G_1 \to G_2$ and $(\sigma_{23}, \alpha_{23}) : G_2 \to G_3$ be two multiplication-commuting Q-fuzzy actions and let

$(\sigma_{13}, \alpha_{13}) = (\sigma_{23}, \alpha_{23}) \circ (\sigma_{12}, \alpha_{12}) : G_1 \to G_3$

Then $(\sigma_{13}, \alpha_{13})$ is a multiplication - commuting Q-fuzzy (right) action of G on H.

Proof.

1. Let (s, x_1, s''), $(s'', x_2, s') \in G_1[G_3, \sigma_{13}\circ d, r]$.

$\alpha_{13}(s, x_1x_2, s')$

$= \sup_t\{\alpha_{12}(\sigma_{23}(d(s)), x_1x_2, t) * \alpha_{23}(s, t, s')\}$

$\geq \sup_{t,t'} \{\alpha_{12}(\sigma_{23}(d(s)), x_1, t') * \alpha_{12}(t', x_2, t) * \alpha_{23}(s, t, s')\}$

$\geq \sup_{t,t'} \{\alpha_{12}(\sigma_{23}(d(s)), x_1, t') * \alpha_{12}(t', x_2, t) * \alpha_{23}(s, t' t'^{-1}t, s')\}$

Since, for all t′ such that $(\sigma_{23}(d(s)), x_1, t')$, $(t', x_2, t) \in G_1[G_2, \sigma_{12}\circ d, r]$ $\sigma_{12} (\sigma_{23}(d(s''))) = r(x_2)$

$\alpha_{12}(t', x_2, t) = \alpha_{12}(t^{-1}t', x_2, d(t))$

$= \alpha_{12}(d(t), x_2^{-1}, t^{-1}t')$

$= \alpha_{12}(\sigma_{23}(d(s')), x_2^{-1}, t^{-1}t')$

and

$\alpha_{23}(s, t' t'^{-1}t, s')\} \geq$

$\geq \alpha_{23}(s, t', s'') * \alpha_{23}(s'', t'^{-1}t, s')$

$\geq \alpha_{23}(s, t', s'') * \alpha_{23}(s', t^{-1} t', s'')$,

it follows that

$\alpha_{13}(s, x_1x_2, s')$

$\geq \sup_{t,t'}\{\alpha_{12}(\sigma_{23}(d(s)), x_1, t') * \alpha_{23}(s, t', s'') * \alpha_{12}(\sigma_{23}(d(s')), x_2^{-1}, t'^{-1}t) * \alpha_{23}(s', t^{-1} t', s'')\}$

$\geq \alpha_{13}(s, x_1, s'') * \alpha_{13}(s', x_2^{-1}, s'')$

$\geq \alpha_{13}(s, x_1, s'') * \alpha_{13}(s'', x_2, s')$

2. Let $(s, x, s') \in G_1[G_3, \sigma_{13}\circ d, r]$.

$\alpha_{13}(s, x, s') =$

$= \sup_t\{\alpha_{12}(\sigma_{23}(d(s)), x, t) * \alpha_{23}(s, t, s')\}$

$\geq \sup_t\{\alpha_{12}(t, x^{-1}, \sigma_{23}(d(s))) * \alpha_{23}(s', t^{-1}, s)\}$

$\geq \sup_t\{\alpha_{12}(d(t), x^{-1}, t^{-1}) * \alpha_{23}(s', t^{-1}, s)\}$

$\geq \sup_t\{\alpha_{12}(\sigma_{23}(d(s')), x^{-1}, t^{-1}) * \alpha_{23}(s', t^{-1}, s)\}$

$= \alpha_{13}(s', x^{-1}, s')$.

3. Let $(s, x, s') \in G_1[G_3, \sigma_{13}\circ d, r]$.

$\alpha_{13}(s, x, s') =$

$= \sup_t\{\alpha_{12}(\sigma_{23}(d(s)), x, t) * \alpha_{23}(s, t, s')\}$

$= \sup_t\{\alpha_{12}(\sigma_{23}(d(s)), x, t) * \alpha_{23}(d(s), t, s^{-1}s')\}$

$= \alpha_{13}(d(s), x, s^{-1}s')$

Let us prove the associativity of composition.

Proposition 3. 2. Let $(\sigma_{12}, \alpha_{12}) : G_1 \to G_2$, $(\sigma_{23}, \alpha_{23}) : G_2 \to G_3$ and $(\sigma_{34}, \alpha_{34}) : G_3 \to G_4$ be three multiplication-commuting Q-fuzzy actions (morphisms). Then

$(\sigma_{34}, \alpha_{34}) \circ ((\sigma_{23}, \alpha_{23}) \circ (\sigma_{12}, \alpha_{12})) =$

$((\sigma_{34}, \alpha_{34}) \circ (\sigma_{23}, \alpha_{23})) \circ (\sigma_{12}, \alpha_{12})$

Proof. Let us denote

$(\sigma_{13}, \alpha_{13}) = (\sigma_{23}, \alpha_{23}) \circ (\sigma_{12}, \alpha_{12}) : G_1 \to G_3$

$(\sigma_{14}, \alpha_{14}) = (\sigma_{34}, \alpha_{34}) \circ (\sigma_{13}, \alpha_{13}) : G_1 \to G_4$

$(\sigma_{24}, \alpha_{24}) = (\sigma_{34}, \alpha_{34}) \circ (\sigma_{23}, \alpha_{23}) : G_2 \to G_4$

$(\sigma'_{14}, \alpha'_{14}) = (\sigma_{24}, \alpha_{24}) \circ (\sigma_{12}, \alpha_{12}) : G_1 \to G_4$

We have

$\sigma_{14} = \sigma_{13} \circ \sigma_{34} = (\sigma_{12} \circ \sigma_{23}) \circ \sigma_{34}$

$= \sigma_{12} \circ (\sigma_{23} \circ \sigma_{34}) = \sigma_{12} \circ \sigma_{24} = \sigma'_{14}$

Let $(y, x, y') \in G_1[G_4, \sigma_{14}\circ d, r]$.

$\alpha_{14}(y, x, y')$

$= \sup_s\{\alpha_{13}(\sigma_{34}(d(y)), x, s) * \alpha_{34}(y, s, y')\}$

$= \sup_s\{\sup_t\{\alpha_{12}(\sigma_{24}(d(y)), x, t) * \alpha_{23}(\sigma_{34}(d(y)), t, s)\} * \alpha_{34}(y, s, y')\}\}$

$= \sup_t\{\alpha_{12}(\sigma_{24}(d(y)), x, t) * \sup_s\{\alpha_{23}(\sigma_{34}(d(y)), t, s)\} * \alpha_{34}(y, s, y')\}\}$

$= \sup_t\{\alpha_{12}(\sigma_{24}(d(y)), x, t) * \alpha_{24}(y, t, y')\}$

$=\alpha'_{14}(y, x, y')$

Hence

$(\sigma_{34}, \alpha_{34}) \circ ((\sigma_{23}, \alpha_{23}) \circ (\sigma_{12}, \alpha_{12})) =$

$= ((\sigma_{34}, \alpha_{34}) \circ (\sigma_{23}, \alpha_{23})) \circ (\sigma_{12}, \alpha_{12})$

Identity morphisms:

For each groupoid G let us denote

$u_G = (id, \mu_G) : G \to G$, where

d is the domain map of G and

$\mu_G: G[G, d, r] \to Q$,

is defined by

$$\mu_G(s, x, t) = \begin{cases} e, \text{if } t = sx \\ 0, \text{if } t \neq sx \end{cases}$$

(e is the neutral element in the quantale Q)

Proposition 3. 3. Let $(\sigma_{12}, \alpha_{12}) : G_1 \to G_2$ be a morphism. Then

$(\sigma_{12}, \alpha_{12}) \circ (\mathrm{id}, \mu_{G_2}) = (\sigma_{12}, \alpha_{12})$

$(\mathrm{id}, \mu_{G_1}) \circ (\sigma_{12}, \alpha_{12}) = (\sigma_{12}, \alpha_{12})$

Proof. Let us denote

$(\sigma, \alpha) = (\sigma_{12}, \alpha_{12}) \circ (\mathrm{id}, \mu_{G_2}) : G_1 \rightharpoonup G_2$

Obviously, $\sigma = \sigma_{12}$ and for all $(s, x, s') \in G_1[G_2, \sigma_{12}\circ d, r]$, we have

$\alpha(s, x, s') =$

$= \sup_t \{\alpha_{12}(d(s), x, t) * \mu_{G_2}(s, t, s')\}$

$= \alpha_{12}(d(s), x, s^{-1}s') * \mu_{G_2}(s, s^{-1}s', s')$

$= \alpha_{12}(s, x, s') * e$

$= \alpha_{12}(s, x, s')$.

Thus, $(\sigma_{12}, \alpha_{12}) \circ (\mathrm{id}, \mu_{G_2}) = (\sigma_{12}, \alpha_{12})$.

Let us denote

$(\sigma', \alpha') = (\mathrm{id}, \mu_{G_1}) \circ (\sigma_{12}, \alpha_{12}): G_1 \rightharpoonup G_2$

Obviously, $\sigma' = \sigma_{12}$ and for all $(s, x, s') \in G_1[G_2, \sigma_{12}\circ d, r]$, we have

$\alpha'(s, x, s') =$

$= \sup_t \{\mu_{G_1}(d(s), x, t) * \alpha_{12}(s, t, s')\}$

$= \mu_{G_1}(d(s), x, x) * \alpha_{12}(s, x, s')$

$= e * \alpha_{12}(s, x, s')$

$= \alpha_{12}(s, x, s')$.

Hence, $(\mathrm{id}, \mu_{G_1}) \circ (\sigma_{12}, \alpha_{12}) = (\sigma_{12}, \alpha_{12})$.

4. MORPHISMS AS FUZZY GROUPOIDS

Let G and H be two groupoids and

$\sigma : H^{(0)} \to G^{(0)}$

be a map. Let us denote

$G\star_\sigma H = \{(g, h), \sigma(d(h)) = d(g) \text{ and } \sigma(r(h)) = r(g)\}$

$G\star_\sigma H$ is a groupoid under the operations

$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$ (iff $d(h_1) = r(h_2)$)

$(g, h)^{-1} = (g^{-1}, h^{-1})$

Since, for the groupoid $G\star_\sigma H$, the range and source maps are

$r(g, h) = (r(g), r(h)) = (\sigma(r(h)), r(h))$

and

$d(g, h) = (\sigma(d(h)), d(h))$,

the unit space of $G\star_\sigma H$ can be identified with $H^{(0)}$ via the mappings: $(v, u)\mapsto u$, $u \mapsto (\sigma(u), u)$.

Example 4. 1. Let us consider two groupoids $G \subset X \times \mathbb{Z} \times X$, $H \subset Y \times \mathbb{Z} \times Y$ and a map $\sigma : Y \to X$. Then

$G\star_\sigma H = \{(\sigma(y), m, \sigma(y'), y, n, y'): (\sigma(y), m, \sigma(y')) \in G \text{ and } (y, n, y') \in H\}$,

that can be identified to

$\{(y, m, n, y'): (y, n, y') \in H, (\sigma(y), m, \sigma(y')) \in G\} \subset Y \times \mathbb{Z} \times \mathbb{Z} \times Y$

Furthermore, according to the results in Section 2 [4], $G\star_\sigma H$ can be identified to

$\{(y, k_{\sigma(y),\sigma(y')}(G) + t\, k_{\sigma(y),\sigma(y)}(G), k_{y,y'}(H) + s\, k_{y,y}(H), y'): (y, y') \in (r,d)(H), (\sigma(y), \sigma(y')) \in (r,d)(G), s, t \in \mathbb{Z}\}$.

Therefore, in this case $G\star_\sigma H$ can be characterized by two families of integers $\{k_{y,y'}(H)\}_{(y, y')}$ and $\{k_{(\sigma(y),\sigma(y'))}(G)\}_{(\sigma(y),\sigma(y'))}$.

Proposition 4. 2. If $\gamma : G\star_\sigma H \to Q$ is a Q-fuzzy subgroupoid of $G\star_\sigma H$, then

$\alpha_\gamma : G[H, \sigma\circ d, r] \to Q$

$\alpha_\gamma(h, g, h') = \gamma(g, h^{-1}h')$,

for all $(h, g, h') \in G[H, \sigma\circ d, r]$, defines a multiplication-commuting Q-fuzzy action (morphism) $(\sigma, \alpha_\gamma) : G \rightharpoonup H$.

Proof.

1. Let (h, g_1, h''), $(h'', g_2, h') \in G[H, \sigma\circ d, r]$.

$\alpha_\gamma(h, g_1g_2, h') = \gamma(g_1g_2, h^{-1}h')$

$= \gamma(g_1g_2, h^{-1} h'' h''^{-1}h')$

$\geq \gamma(g_1, h^{-1} h'') * \gamma(g_2, h''^{-1}h')$

$= \alpha_\gamma(h, g_1g_2, h'') * \alpha_\gamma(h'', g_1g_2, h')$

2. If $(h, g, h') \in G[H, \sigma\circ d, r]$, then

$\alpha_\gamma(h, g, h') = \gamma(g, h^{-1}h')$

$\geq \gamma(g^{-1}, h'^{-1}h)$

$= \alpha_\gamma(h', g^{-1}, h)$.

Proposition 4.3. If $(\sigma, \alpha) : G \rightharpoonup H$ a multiplication-commuting Q-fuzzy action (morphism), then

$\gamma_{(\sigma, \alpha)} : G\star_\sigma H \to Q$

$\gamma_{(\sigma, \alpha)}(g, h) = \alpha(r(h), g, h)$

is a Q-fuzzy subgroupoid of $G\star_\sigma H$.

Proof.

1. If $((g_1, h_1), (g_2, h_2)) \in G\star_\sigma H^{(2)}$, then

$\gamma_{(\sigma, \alpha)}(g_1g_2, h_1h_2) = \alpha(r(h_1), g_1g_2, h_1h_2)$

$\geq \alpha(r(h_1), g_1, h_1) * \alpha(h_1, g_2, h_1h_2)$

$= \alpha(r(h_1), g_1, h_1) * \alpha(d(h_1), g_2, h_2)$

$= \alpha(r(h_1), g_1, h_1) * \alpha(r(h_2), g_2, h_2)$

$= \gamma_{(\sigma, \alpha)}(g_1, h_1) * \gamma_{(\sigma, \alpha)}(g_2, h_2)$.

2. If $(g, h) \in G\star_\sigma H$, then

$\gamma_{(\sigma, \alpha)}(g, h) = \alpha(r(h), g, h)$

$\geq \alpha(h, g^{-1}, r(h))$

$= \alpha(d(h), g^{-1}, h^{-1})$

$= \gamma_{(\sigma, \alpha)}(g^{-1}, h^{-1})$

5. CONNECTION WITH THE Q-REL CATEGORY

The category Q-Rel has sets as objects and fuzzy Q-valued relations as morphisms, with composition defined via quantale joins and multiplication.

Let us recall that, given sets X and Y, a Q-valued fuzzy relation is a function

$R : X \times Y \to Q$

interpreting R(x,y) as the degree to which x is related to y. Given $R : X \times Y \to Q$ and $S : Y \times Z \to Q$, their composite is $S \circ R : X \times Z \to Q$ defined by

$S \circ R\,(x, z) = \sup\{R(x, y) * S(y, z), y \in Y\}$

The identity on a set X is $1_X : X \times X \to Q$

$$1_X(x, x') = \begin{cases} e, \text{ if } x = x' \\ 0, \text{ if } x \neq x' \end{cases}$$

Let us consider the following subcategory of Q-Rel, that we call Q-ARel

Objects: sets endowed with groupoid structure.

Morphisms: fuzzy Q-valued relations $R : G \times H \to Q$, with the property that G and H are groupoids and there is a map $\sigma : H^{(0)} \to G^{(0)}$ such that $d(h_1) = r(h_2)$

1. $R(g, h) = 0$ if $(g, h) \notin G\star_\sigma H$
2. $R(g_1g_2, h_1h_2) \geq R(g_1, h_1) * R(g_1, h_1)$ for all $(g_1, h_1), (g_2, h_2) \in G\star_\sigma H$ such that $d(h_1) = r(h_2)$.
3. $R(g^{-1}, h^{-1}) \geq R(g, h)$ for all $(g, h) \in G\star_\sigma H$

Let us call a relation Q-valued relations $R : G \times H \to Q$ that satisfies the above conditions 1, 2 and 3 multiplicative-commuting Q-fuzzy relation.

Let us remark that if $R: G_1 \times G_2 \to Q$ and $S: G_2 \times G_3 \to Q$ are multiplicative-commuting Q-fuzzy relations, then $S \circ R$ is a multiplicative-commuting Q-fuzzy relation. Indeed, let $\sigma_R : G_2^{(0)} \to G_1^{(0)}$ the map associated to R and $\sigma_S : G_3^{(0)} \to G_2^{(0)}$ the map associated to S and let $\sigma = \sigma_{S\circ R} = \sigma_R \circ \sigma_S$. If $(x, s) \notin G_1\star_\sigma G_3$, then for every $t \in G_2$, $(x, t) \notin G_1\star_{\sigma_R} G_2$ or $(t, s) \notin G_2\star_{\sigma_S} G_3$. Thus, if $(x, s) \notin G_1\star_\sigma G_3$, $\sigma_{S\circ R}(s, t) = 0$.

Let $(x_1, s_1), (x_2, s_2) \in G_1\star_\sigma G_3$ such that $d(s_1) = r(s_2)$. Then

$\{t: (x_1x_2, t) \in G_1\star_{\sigma_R} G_2,\ (t, s_1s_2) \in G_2\star_{\sigma_S} G_3\}$
$\supset \{ t't'^{-1}t, t, t' \in G_2$, such that $(x_1, t') \in G_1\star_{\sigma_R} G_2$ and $(t', s_1) \in G_2\star_{\sigma_S} G_3\}$.

Hence,

$S \circ R(x_1x_2, s_1s_2) \geq \sup_t\{R(x_1x_2, t) * S(t, s_1s_2)\}$
$\geq \sup_{t,t'}\{R(x_1x_2, t't'^{-1}t) * S(t't'^{-1}t, s_1s_2)\}$
$\geq \sup_{t,t'}\{R(x_1, t') * R(x_2, t'^{-1}t) * S(t', s_1) * S(t'^{-1}t, s_2)\}$
$\geq \sup_{t'}\{R(x_1, t') * S(t', s_1)\} * \sup_{t''}\{R(x_2, t'')\} * S(t'', s_2)\}$
$\geq S \circ R(x_1, s_1) * S \circ R(x_2, s_2)$

If $(x, s) \in G_1\star_\sigma G_3$, then

$S \circ R(x^{-1}, s^{-1})$
$= \sup_t\{R(x, t^{-1}) * S(t^{-1}, s)\} = S \circ R(x, s)$.

Obviously, $1_G : G \times G \to Q$ is a multiplicative-commuting Q-fuzzy relation with respect to $\sigma = id: G^{(0)} \to G^{(0)}$.

Proposition 5. 1. The category GrpFAct is isomorphic to the subcateqory Q-ARel of Q-Rel.

Proof. Let us define the following functors:

1. $F_1 :$ GrpFAct $\to$ QARel
$G \mapsto G$, $F_1((\sigma, \alpha)) = R_{(\sigma, \alpha)}$,
where if $(\sigma, \alpha) : G \to H$ is a multiplication - commuting fuzzy action of G on H, then $R_{(\sigma, \alpha)} : G \times H \to Q$ is the Q-valued fuzzy relation defined by
$$R_{(\sigma, \alpha)}(g, h) = \begin{cases} \alpha(r(h), g, h), \text{ if } (g, h) \in G\star_\sigma H \\ 0, \quad \text{otherwise} \end{cases}$$
that is a multiplicative-commuting Q-fuzzy relation.
2. $F_2 :$ QARel $\to$ GrpFAct
$G \mapsto G$, $F_2(R) = (\sigma_R, \alpha_R)$.
where if R is a multiplicative-commuting Q-fuzzy relation and $\sigma_R : H^{(0)} \to G^{(0)}$ is he map associated to R, then $\alpha_R : G[H, \sigma, r] \to Q$ is defined by
$\alpha_R(h, g, h') = R(g, h^{-1}h')$
for all $(h, x, h') \in G[H, \sigma, r]$. It is easy to check that (σ_R, α_R) is multiplication - commuting Q - fuzzy action of G on H.

If $(g, h) \in G\star_\sigma H$,

$F_1(F_2(R))\,(g, h) = \alpha_R(r(h), g, h) = R(g, h)$.

Moreover, for $(h, x, h') \in G[H, \sigma, r]$,

$F_2(F_1(\sigma, \alpha))\,(h, g, h') = F_2(R_{(\sigma, \alpha)})\,(h, g, h') =$
$= R_{(\sigma, \alpha)}(g, h^{-1}h') = \alpha(r(h^{-1}h'), g, h^{-1}h')$
$= \alpha(d(h), g, h^{-1}h') = \alpha(h, g, h')$

Since $F_1F_2 = 1_{QARel}$ and $F_2F_1 = 1_{GrpFAct}$ it follows that GrpFAct and Q-ARel are isomorphic categories.

6. CONCLUSION

In this paper we introduced a category whose objects are groupoids and whose morphisms are Q-fuzzy actions compatible with the groupoid structure. This categorical viewpoint not only clarifies the compositional nature of fuzzy actions compatible with the groupoid structure but also opens the door to further abstractions

By showing that these morphisms can be realized as Q-fuzzy subgroupoids of a suitably chosen ambient groupoid, we provided a structural interpretation that clarifies the nature of Q-graded fuzzy behavior within groupoids.

Moreover, the categorical isomorphism with a subcategory of Q-Rel demonstrates that the proposed framework fits naturally within the broader relational semantics induced by a unital quantale.

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